Sequences in Metric Spaces

Definition Sequences

A sequence in a metric space (X,d) is an element of $X^{\mathbb{N}}$

Here

$x = (x_1, x_2, \ldots, x_n, \ldots) \in X^{\mathbb{N}}$

$x_i \in X$

1) Order is important
2) Not a set
3) Nothing to stop $x_i = x_j$, $i \neq j$

Alternatively it can be said a sequence is a function

$$f : \mathbb{N} \to X$$

$$f(n) = x_n \qquad x_n \in X$$

Notation: Sequences represented by

$$(x_n)_{n\geq 1}, \quad (x_n)_{n\in\mathbb{N}}, \quad \{x_n\}_{n\geq 1}, \quad \{x_n\}_{n\in\mathbb{N}}$$

It can also be written as

$$(x_n)_{n=1}^{\infty}, \quad \{x_n\}_{n=1}^{\infty}$$

Convergence of Sequences

The main issue with sequence is convergence.

Definition Convergence

Let $(x_n)_{n=1}^{\infty}$ be a sequence in (X,d).

Then $(x_n)_{n=1}^{\infty}$ converges to $x \in X \iff$ for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $d(x_n, x) < \varepsilon$ for all $n > N$

If this case, we write

$$x_n \to x \quad \text{as} \quad n \to \infty$$

If no such x exists

$$x_n \not\to x \quad \text{as} \quad n \to \infty$$

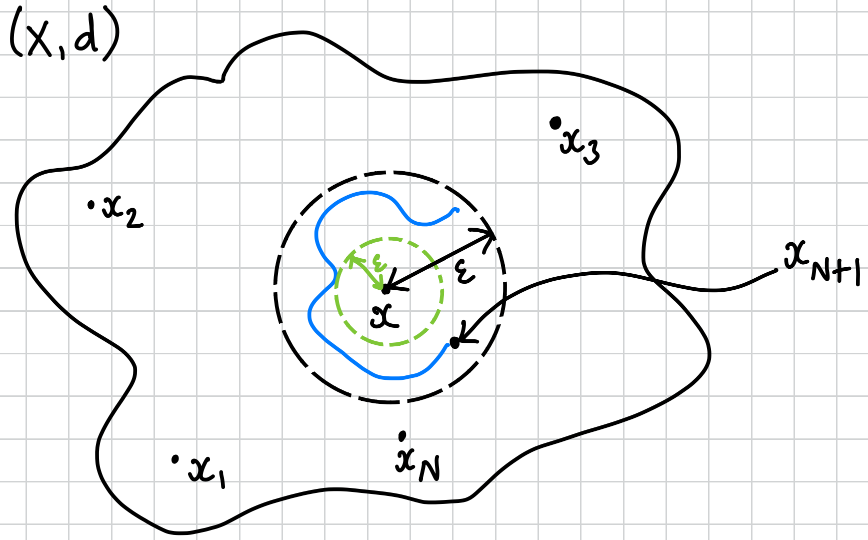
then (x_n) is divergent

The definition can be recast in terms of open balls

$$\boxed{x_n \to x \text{ as } n\to\infty \iff \forall\, \varepsilon>0\ \exists\, N=N(\varepsilon) \text{ such that } x_n \in B(x,\varepsilon)\ \forall n>N}$$

open ball contains all but finitely many x_n

$n=1,\dots,N$



Useful equivalence of convergence

$\lim_{n\to\infty} d(x_n,x)=0$

$$x_n \to x \iff d(x_n,x)\to 0 \text{ as } n\to\infty$$

Convergence in $\mathbb{R}^N$

In $\mathbb{R}^N$, $N\in\mathbb{N}$ and metrics d_1, d_2, d_∞ convergence is equivalent to simultaneous componentwise convergence

Notation: $\underline{x} = (x_1,\dots,x_N)$ where $\underline{x}\in\mathbb{R}^N$

Let $\{\underline{x}_n\}_{n\geq 1}$ be a sequence in $\mathbb{R}^N$

$$\underline{x}_n = (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})$$

Theorem: Component-wise convergence in $(\mathbb{R}^N, d_\infty)$

In $(\mathbb{R}^N, d_\infty)$, convergence is equivalent to simultaneous component-wise convergence.

$$\underline{x}_n \to \underline{x} \text{ as } n\to\infty \iff x_n^{(i)} \to x_i \text{ as } n\to\infty$$
$$\forall\, i\in\{1,\dots,N\}$$

Proof:

($\Longrightarrow$): Suppose $\underline{x}_n \to \underline{x}$ as $n\to\infty$

Then, $\forall\,\varepsilon>0$, $\exists\, N=N(\varepsilon)$ such that $\forall\, n>N$,

$$d_\infty(\underline{x}_n, \underline{x}) < \varepsilon$$

$$d_\infty(\underline{x}_n, \underline{x}) < \varepsilon \implies \max\{|x_n^{(i)} - x_i| : 1 \le i \le N\} < \varepsilon$$
$$\implies |x_n^{(i)} - x_i| < \varepsilon \quad \text{for any } i \in \{1, \dots, N\} \quad \text{for any } n > N$$

if it holds for the maximum, it holds for any one in particular.

This means that real sequence $(x_n^{(i)})$ converges to x_i

$$x_n^{(i)} \to x_i \quad \text{as} \quad n \to \infty$$

$(\Leftarrow)$: For each $i \in \{1, \dots, N\}$, the sequence $(x_n^{(i)})$ convergent to x_i

Want to show that $\underline{x}_n \to \underline{x}$ as $n \to \infty$

Then for any $i \in \{1, \dots, N\}$, $\exists N_i > 0$ such that

$$|x_n^{(i)} - x_i| < \varepsilon \qquad \forall n > N_i$$

Drawing the diagram

$$\underline{x}_1 = (x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, \dots, x_1^{(N)})$$
$$\underline{x}_2 = (x_2^{(1)}, x_2^{(2)}, x_2^{(3)}, \dots, x_2^{(N)})$$
$$\underline{x}_3 = (x_3^{(1)}, x_3^{(2)}, x_3^{(3)}, \dots, x_3^{(N)})$$
$$\vdots$$

Let $N := \max\{N_1, \dots, N_N\}$

Then $|x_n^{(i)} - x_i| < \varepsilon \quad \forall n > N$ and each $i \in \{1, \dots, N\}$

At each such $n > N$, at least one of the terms $|x_n^{(i)} - x_i|$ is maximal but this means that

$$d_\infty(\underline{x}_n, \underline{x}) < \varepsilon \quad \text{for each } n > N$$

$\blacksquare$

Theorem Component-wise convergence in $(\mathbb{R}^N, d_p)$, $p \geq 1$

In $(\mathbb{R}^N, d_p)$ convergence is equivalent to simultaneous component-wise convergence

$$\underline{x}_n \to \underline{x} \text{ as } n \to \infty \iff x_n^{(i)} \to x_i \text{ as } n \to \infty$$

$$\forall\, i \in \{1, \dots, N\}$$

Proof:

($\Rightarrow$): Suppose $\underline{x}_n \to \underline{x}$ as $n \to \infty$

Then for any $\varepsilon > 0$ $\exists N = N(\varepsilon)$ such that $\forall n > N$,

$$d_p(x, y) < \varepsilon \implies \left(\sum_{i=1}^{N} |x_n^{(i)} - x_i|^p \right)^{1/p} < \varepsilon$$

$$\implies |x_n^{(i)} - x_i| < \left(\sum_{i=1}^{N} |x_n^{(i)} - x_i|^p \right)^{1/p} < \varepsilon$$

$$\implies |x_n^{(i)} - x_i| < \varepsilon \qquad \forall n > N$$

($\Leftarrow$): Suppose that $x_n^{(i)} \to x_i$ as $n \to \infty$ for each $i \in \{1, \dots, N\}$

Then for each i and any $\varepsilon > 0$, $\exists N_i = N_i(\varepsilon)$ such that $\forall n > N$

$$|x_n^{(i)} - x_i| < \frac{\varepsilon}{n^{1/p}} \implies |x_n^{(i)} - x_i|^p < \frac{\varepsilon^p}{n}$$

$$\implies \sum_{i=1}^{N} |x_n^{(i)} - x_i|^p < \varepsilon^p$$

$$\implies \left(\sum_{i=1}^{N} |x_n^{(i)} - x_i|^p \right)^{1/p} < \varepsilon$$

$$\implies d_p(\underline{x}_n, \underline{x}) < \varepsilon$$

$\blacksquare$

Function sequences

Consider $X \subseteq \mathbb{R}$. If to every $n = 1,2,\dots$, is assigned a real valued function f_n,

$$\boxed{(f_n)_{n\geq 1} \text{ is a function sequence in } X}$$

Definition Pointwise convergence of functions

Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions $f_n : X \to Y$. The function f is the pointwise limit of sequence $f_n \iff$ for any $x_0 \in X$, $\lim_{n\to\infty} f_n(x_0) = f(x_0)$ (take x_0 and fix it)

In which case we say that f_n converges to f pointwise

$$f_n \xrightarrow{pt} f$$

The ε-δ definition for pointwise convergence

$\lim_{n\to\infty} f_n(x) \to f(x) \iff$ given $\varepsilon > 0$ and $x \in X$ $\exists\, N = N(x,\varepsilon) \in \mathbb{N}$ s.t $\forall n > N$,

$$|f(x) - f_n(x)| < \varepsilon$$

(here, take an $x \in X$ and fix it, check $\lim_{n\to\infty} f_n(x) = f(x)$)

Uniform convergence

A function converges uniformly, we can find a single ε that works for all $x \in X$ and therefore

$$N = N(\varepsilon)$$

no dependance on x

ε-δ definition of uniform convergence

$f_n \to f$ uniformly $\iff$ given $\varepsilon > 0$, $\exists\, N = N(\varepsilon) \in \mathbb{N}$ s.t $\forall n > N$

$$|f(x) - f_n(x)| < \varepsilon \quad \forall x \in X$$

Example: To show that different metrics on the same set can have different convergent sequences

Consider function sequence $(f_n)_{n=1}^{\infty}$ where $f_n \in C[0,1]$ and

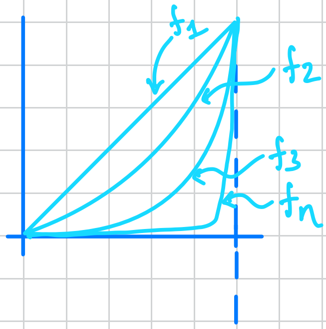
space of continuous function on $[0,1]$

$$f_n : [0,1] \to \mathbb{R};\ t \mapsto t^n$$

Ask about convergence with respect to n

i) d_2 metric $\quad d_2(f,g) = \left(\int_0^1 (f(t)-g(t))^2 \, dt \right)^{1/2}$

ii) d_∞ metric $\quad d_\infty(f,g) = \sup\{ |f(t)-g(t)| : t \in [0,1] \}$



i) Claim: $f_n \to 0$ as $n \to \infty$

Evaluate $d_2(f_n, 0)$:

$$d_2(f_n,0) = \left(\int_0^1 (f_n(t) - 0)^2 \, dt \right)^{1/2}$$

$$= \left(\int_0^1 (t^n)^2 \, dt \right)^{1/2}$$

$$= \left(\int_0^1 t^{2n} \, dt \right)^{1/2}$$

$$= \sqrt{ \left[\frac{1}{2n+1} t^{2n+1} \right]_0^1 } = \sqrt{\frac{1}{2n+1}} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

$\therefore f_n \to 0$ as $n \to \infty$ (note $0 \in C([0,1])$)

ii) Does $f_n \to 0$ as $n\to\infty$ if we are in $(C[0,1], d_\infty)$

Evaluate $d_\infty(f_n, 0)$:

$$\begin{aligned} d_\infty(f_n,0) &= \sup\{|f_n(t)-0| : t\in[0,1]\} \\ &= \sup\{|f_n(t)| : t\in[0,1]\} \\ &= \sup\{t^n : t\in[0,1]\} \\ &= 1 \not\to 0 \text{ as } n\to\infty \end{aligned}$$

Theorem

Suppose (X,d) and $(X,\tilde{d})$ are equivalent $\exists \lambda>0$ such that $1/\lambda\, \tilde{d}(x,y) \le d(x,y) \le \lambda \tilde{d}(x,y)$

$(x_n)_{n=1}^{\infty}$ converges to x in (X,d)

$$\Updownarrow$$

$(x_n)_{n=1}^{\infty}$ converges to x in $(X,\tilde{d})$

__Proof__:

Suppose that $x_n \to x$ as $n\to\infty$ in (X,d)

Let $\varepsilon>0$ be given and set $\tilde{\varepsilon}=\varepsilon/\lambda$

Then $\exists N>0$ such that $d(x,x_n) < \tilde{\varepsilon} = \dfrac{\varepsilon}{\lambda}$ $\quad \forall n>N$

But

$$\frac{1}{\lambda}\tilde{d}(x_n,x) < d(x_n,x) < \tilde{\varepsilon} = \frac{\varepsilon}{\lambda}$$

and therefore

$$\tilde{d}(x_n,x) \le \lambda d(x_n,x) < \lambda\tilde{\varepsilon} = \cancel{\lambda}\frac{\varepsilon}{\cancel{\lambda}} = \varepsilon$$

$$\Rightarrow \tilde{d}(x_n,x) < \varepsilon$$

$(\Leftarrow)$: Suppose $x_n \to x$ with respect to $\tilde{d}$.

Let $\varepsilon>0$ be given and set $\hat{\varepsilon}=\varepsilon/\lambda$.

Then $\exists N>0$ such that $\tilde{d}(x_n,x) < \dfrac{\varepsilon}{\lambda}$ $\quad \forall n>N$

That is $\lambda\tilde{d}(x_n,x) < \varepsilon \quad \forall n>N$

But $d(x_n,x) \le \lambda\tilde{d}(x_n,x) < \varepsilon \quad \forall n>N$ ∎

Uniqueness of Limits

Theorem: Uniqueness of Limits

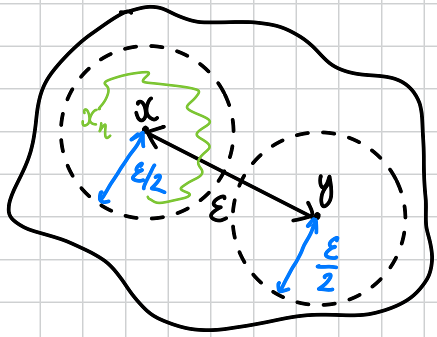
Let (X,d) be a metric space

Suppose $x_n \to x$ as $n \to \infty$ and $x_n \to y$ as $n\to\infty$. Then

$$x=y$$

That is limit of convergent sequences are unique

Proof: (uniqueness proofs: use contradiction):



Lets assume $x \neq y$.

Thus $d(x,y) = \varepsilon > 0$ and set $\varepsilon = \delta/2$

As $x_n \to x$ as $n\to\infty$ $\exists N = N(\delta) > 0$ such that

$$d(x_n, x) < \delta = \varepsilon/2 \quad \forall n > N$$

Similarly since $x_n \to y$ as $n \to \infty$, $\exists \hat{N} = \hat{N}(\delta) > 0$ such that

$$d(x_n, y) < \delta = \varepsilon/2 \quad \forall n > N$$

Set $M = \max\{N, \hat{N}\}$. Then both conditions hold, i.e.

$$d(x_n, x) < \varepsilon \quad \text{AND} \quad d(x_n, y) < \varepsilon \quad \forall n > M$$

By triangle inequality,

$$\begin{aligned} d(x,y) &\leq d(x, x_n) + d(x_n, y) \\ &< \delta + \delta \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall n > M \end{aligned}$$

But $d(x,y) = \varepsilon \implies$ contradiction

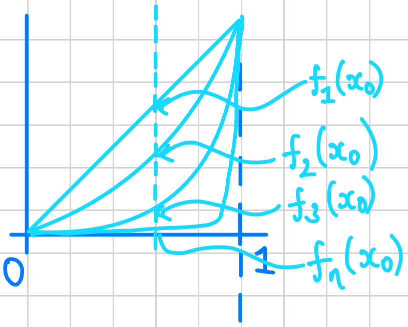
Therefore $x = y$

$\blacksquare$

Example: Example of pointwise convergence of functions

Consider sequence of functions

$$f_n(t) = t^n \quad \text{defined on } t \in [0,1]$$



Q) Does it converge

Q) What happens if you vary x_0

$$f_n \xrightarrow{pt} f \quad \text{as} \quad n \to \infty$$

where

$$f(t) = \begin{cases} 0 & t \neq 1 \\ 1 & t = 1 \end{cases}$$

__Proof__:

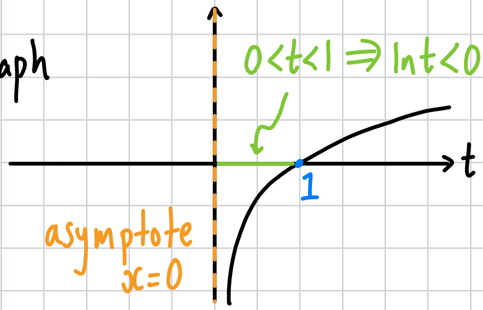
1) CASE 1: If $t=0$, $f_n(0) = 0^n = 0$, $|f_n(0) - 0| = |0-0| = 0 < \varepsilon$ ✓

2) CASE 2: If $t=1$, $f_n(1) = 1^n = 1$, $|f_n(1) - 1| = |1-1| = 0 < \varepsilon$ ✓

3) CASE 3: If $0<t<1$, we claim that $f_n(t) = t^n \to 0$ as $n \to \infty$

$$|f_n(t) - 0| = |t^n - 0| = |t^n| < \varepsilon \implies t^n < \varepsilon$$

ln(t) graph



$$\implies n\ln(t) < \ln(\varepsilon) \quad \text{applying } \log_e$$

$$\implies n > \frac{\ln(\varepsilon)}{\ln(t)} \quad \ln(t)<0 \text{ for } 0<t<1$$

Therefore choose $N > \dfrac{\ln(\varepsilon)}{\ln(t)}$ Archimedean property

Then for any $n>N$,

$$|f_n(t) - 0| = |t^n - 0| = t^n$$

Since $n > N > \dfrac{\ln(\varepsilon)}{\ln(t)} \implies n > \dfrac{\ln(\varepsilon)}{\ln(t)}$

$$\implies n\ln(t) < \ln(\varepsilon) \implies \ln(t^n) < \ln(\varepsilon)$$

$$\implies t^n < \varepsilon \quad \text{(exponentiating both sides)} \quad \blacksquare$$

Note: Our abstract notion of convergence of a sequence contains the classical notion

$(\mathbb{R}, d_1)$, $d_1(x,y) = |x-y|$

Also contains

$(\mathbb{R}^2, d_2)$ and $(\mathbb{C}, d_2)$ $d_2(z,z') = |z-z'|$

Note: Series $(\mathbb{R}, d)$

Recall we are often concerned with sums that have infinite terms;

$$S_\infty = \sum_{n=1}^{\infty} x_n$$

where S_∞ is the limit (if it exists) of the sequence

$$S_N = \sum_{n=1}^{N} x_n \qquad \text{(partial sums)}$$

We want $S_\infty = \lim_{N\to\infty} S_N$

This can be moved to an abstract metric space if (X,d) has a notion of addition.

(may not in general)

Cauchy Sequences

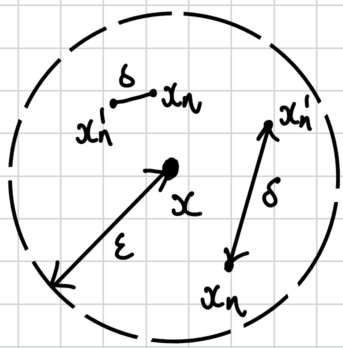
Cauchy sequences deal with closeness of terms

Definition Cauchy Sequences

Suppose (X,d) is a metric space and $(x_n)_{n=1}^{\infty}$ a sequence in X.

(x_n) is Cauchy $\iff \forall \varepsilon > 0, \exists N = N(\varepsilon) > 0$ such that

$$d(x_m, x_n) < \varepsilon \quad \forall m,n > N$$



Remark: Nowhere in the definition of Cauchy do we assume that $x_n \to x$.

No such x may exist. Cauchy sequence need not be convergent

Example: Cauchy sequence that is not convergent

Take any sequence of rational numbers p_n/q_n for which $(p_n/q_n)^2 \to 2$ as $n \to \infty$

$\mathbb{Q} = \{m/n : m \in \mathbb{Z},\ n \in \mathbb{N}\}$

p_n/q_n is Cauchy sequence but no x exists in $\mathbb{Q}$ such that $\lim_{n\to\infty} \frac{p_n}{q_n} = x$ $\sqrt{2} \notin \mathbb{Q}$

Convergent $\Rightarrow$ Cauchy

Theorem Convergent $\Rightarrow$ Cauchy

Let (X,d) be a metric space and $(x_n)_{n=1}^{\infty}$ be a convergent sequence to x. Then $(x_n)_{n=1}^{\infty}$ is Cauchy

Proof: Let $\varepsilon > 0$ be given. Set $\delta = \frac{\varepsilon}{2}$

Then $\exists N = N(\delta) > 0$ such that $d(x_n, x) < \delta = \frac{\varepsilon}{2}$ $\forall n > N$

But now $d(x_n, x_m)$ where $n > N$ satisfies

Δ-inequality: $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$

$$< \delta + \delta = \varepsilon$$

So $(x_n)_{n=1}^{\infty}$ is Cauchy ■

Another equivalent defn for Cauchy

$(x_n)_{n=1}^{\infty}$ is Cauchy $\Longleftrightarrow$ $d(x_m, x_n) \to 0$ as $m, n \to \infty$

Example of a Cauchy function sequence

Consider space $C([0,1])$, the sequence $f_1, f_2, f_3, \ldots$ given by

$$f_n(x) = \frac{nx}{n+x}$$

with uniform metric

$$d_\infty(f,g) = \sup\{|f(x) - g(x)$$

Therefore calculating $d_\infty(f_m, f_n)$

$$d_\infty(f_m, f_n) = \sup\{|f_m(x) - f_n(x)| : x \in [0,1]\}$$

$$= \sup\left\{ \left| \frac{mx}{m+x} - \frac{nx}{n+x} \right| : x \in [0,1] \right\}$$

$$= \sup\left\{ \frac{(m-n)x^2}{(m+x)(n+x)} : x \in [0,1] \right\}$$

extreme value thm

Since $\frac{(m-n)x^2}{(m+x)(n+x)}$ is continuous on $[0,1]$, it has a maximum at some $x_0 \in [0,1]$

Therefore

$$d_\infty(f_m, f_n) = \frac{(m-n)x_0^2}{(m+x_0)(n+x_0)} \leq \frac{x_0^2}{n+x_0} \leq \frac{1}{n} \to 0$$

$$\Rightarrow d_\infty(f_m, f_n) \to 0 \text{ as } n \to \infty$$

$$\Rightarrow (f_n)_{n=1}^{\infty} \text{ is Cauchy}$$

■

Therefore in general,

- Cauchy $\not\Rightarrow$ Convergence
- Convergence $\Rightarrow$ Cauchy (*)

We can use contrapositive of (*) to get a test for non-convergence (divergence test)

Divergence Test

not Cauchy $\Longrightarrow$ not convergent

Example: Showing harmonic series is divergent

Work in $(\mathbb{R}, d)$, $d(x,y) = |x-y|$

Harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

We need to show

$$S_N = \sum_{n=1}^{N} \frac{1}{n}$$

is not Cauchy, i.e. it satisfies the negation of definition of Cauchy

$\exists\, \varepsilon > 0$ s.t $\forall N = N(\varepsilon) > 0$, $\exists m, n > N$ $d(x_m, x_n) \geq \varepsilon$

Take $m = 2n$

$$a_{2n} - a_n = \frac{1}{2N} + \frac{1}{2N-1} + \cdots + \frac{1}{N+1} + \cancel{\frac{1}{N}} + \cancel{\cdots} + \cancel{1} - \cancel{\frac{1}{N}} - \cancel{\frac{1}{N-1}} - \cdots - \cancel{1}$$

$$= \frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{2N}$$

$$\geq \frac{1}{N+N} + \frac{1}{N+N} + \cdots + \frac{1}{N+N}$$

$$= \frac{\cancel{N}}{2\cancel{N}} = \frac{1}{2}$$

$$\Rightarrow |a_{2n} - a_n| \geqslant \frac{1}{2}$$

So take $\varepsilon > 0 = \frac{1}{2}$, $\forall N = N(\varepsilon)$, $\exists\, m = 2n,\ n \geq N$ such that

$$|a_{2N} - a_N| \geq \varepsilon = \frac{1}{2}$$

$\Rightarrow$ S_N is not Cauchy

$\Rightarrow$ S_N is not convergent ◀

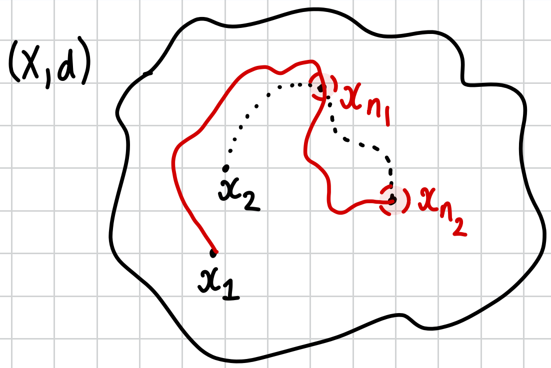
Subsequence

Given (X,d) and a sequence in $(x_n)_{n=1}^{\infty}$ in X.

Definition Subsequences

A subsequence of (x_n) is a sequence of elements $x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k} \cdots$ where $n_i \in \mathbb{N}$ and $n_1 < n_2 < \cdots < n_k < \cdots$

$$(x_{n_k})_{k=1}^{\infty}$$



$n_k \to \infty$ as $k \to \infty$

Lemma

For any increasing sequence $(n_k)_{k \geq 1}$,

$$n_k \geq k$$

Proof: using induction

For $k=1$, $n_1 \geq 1$ which is trivially true

Assuming true for k,

$$n_k \geq k \qquad \text{inductive hypothesis}$$

Showing $P(k) \Rightarrow P(k+1)$

Since $(n_k)_{k\geq 1}$ is an increasing sequence,

$$n_{k+1} > n_k$$

Further by inductive hypothesis,

$$n_k \geq k \Rightarrow n_k + 1 \geq k+1$$
$$\Rightarrow n_{k+1} \geq n_k + 1 \geq k+1$$
$$\Rightarrow n_{k+1} \geq k+1$$

■

Almost immediately, $x_n \to x$ as $n \to \infty \Rightarrow$ any subsequence x_{n_k} converges to x

Theorem Convergence $\Rightarrow$ every subsequence converges to same limit

Suppose (X, d) is a metric space and $(x_n)_{n=1}^{\infty}$ is a sequence in X.

If $x_n \to x$ as $n \to \infty$ and $(a_{n_k})_{k\in\mathbb{N}}$ is a subsequence, then

$$x_{n_k} \to x \quad \text{as} \quad k \to \infty$$

Proof:

$x_n \to x$ as $n \to \infty \Rightarrow$ given any $\varepsilon > 0$, $\exists N = N(\varepsilon)$ s.t $\forall n > N$

$$d(x_n, x) < \varepsilon$$

Since (n_k) is a strictly increasing sequence of natural numbers, $n_k \geq k$.

It follows that $k > N \Rightarrow n_k > N$

$$\Rightarrow d(x_{n_k}, x) < \varepsilon$$
$$\Rightarrow x_{n_k} \to x \quad \text{as} \quad k \to \infty$$

■

We can use the contrapositive of the above theorem for another divergence test contrapositive test for divergence using subsequence

$$(x_n)_{n=1}^{\infty} \in X^{\mathbb{N}}$$

if $\exists\, n_1 < n_2 < \cdots < n_k < \cdots$ & $n_1' < n_2' < \cdots < n_k' < \cdots$

such that $x_{n_i} \to x$ as $i \to \infty$ & $x_{n_i'} \to y$ as $i \to \infty$

with $x \neq y \Rightarrow (x_n)_{n=1}^{\infty}$ is divergent.

Classic example: $x_n = (\pm 1)^n$ in $\mathbb{R}$

For even terms $n = 2K$, $n_1 = 2$, $n_2 = 4$, $n_3 = 6$, $\cdots \rightsquigarrow (1, 1, \ldots, 1) \Rightarrow x_{2K} \to 1$

For odd terms $n = 2K+1$, $n_1' = 1$, $n_2' = 3$, $n_3' = 5$, $\cdots \rightsquigarrow (-1, -1, \ldots, -1) \Rightarrow x_{2K+1} \to -1$

$$\lim_{K \to \infty} x_{2K} = 1, \qquad \lim_{K \to \infty} x_{2K+1} = -1$$

Theorem

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ a Cauchy sequence and $(x_{n_K})_{K=1}^{\infty}$ a convergent subsequence.

Then $(x_n)_{n=1}^{\infty}$ is convergent and

$$\lim_{n \to \infty} x_n = \lim_{K \to \infty} x_{n_K} = x$$

Proof: We need to show that for any $\varepsilon > 0$, $\exists\, N = N(\varepsilon)$ s.t $d(x_n, x) < \varepsilon$ $\forall n > N$

Let $\varepsilon > 0$ be given.

By definition of Cauchy, $\exists\, N_1 = N_1(\varepsilon) > 0$ such that $d(x_n, x_m) < \varepsilon$ $\forall m, n > N_1$

By definition of convergence, $\exists\, N_2 = N_2(\varepsilon) > 0$ such that $d(x_{n_K}, x) < \frac{\varepsilon}{2}$ $\forall K > N_2$

Choose $N = \max\{N_1, N_2\}$. Thus both conditions hold at the same time.

By triangle inequality

$$\begin{aligned} d(x_n, x) &\leq d(x, x_{n_K}) + d(x_{n_K}, x_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad (n_K \geq K) \\ &= \varepsilon \end{aligned}$$

$\blacksquare$

Upshot: the main obstacle for a Cauchy sequence convergence seems to be non-existence of the should be limit

Example of a function sequence that is Cauchy but not convergent

Consider metric space $(C([0,1]), d_2)$ where

$$d_2(f,g) = \int_0^1 |f(x)-g(x)| \, dx \qquad f,g \in C[0,1]$$

Consider function sequence $\{f_n\}_{n\geq 2}$ be a sequence defined by

$$f_n(x) = \begin{cases} 0 & 0 \leq x \leq 1/2 - 1/n \\ n\left(x-\frac{1}{2}\right)+1 & 1/2 - 1/n < x \leq 1/2 \\ 0 & 1/2 \leq x \leq 1 \end{cases}$$

Clearly $f_n \in C[0,1]$. This is Cauchy because

remember trick

$$d_2(f_m,f_n) = \int_0^1 |f_n(x) - f_m(x)| \, dx \leq \int_{1/2-1/m}^{1/n} f_m(x)\,dx + \int_{1/2-1/n}^{1/2} f_n(x)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{m}+\frac{1}{n}\right) \to 0 \quad \text{as } m,n \to \infty$$

$$\Rightarrow d(f_m,f_n) \to 0 \quad \text{as } m,n\to\infty$$

$$\Rightarrow \text{Cauchy}$$

Now suppose $f_n \to f$, i.e. $d(f_n,f) \to 0$ as $n \to \infty$

$$d(f_n,f) = \underbrace{\int_0^{1/2-1/n} |0-f(x)|\,dx}_{(1)} + \int_{1/2-1/n}^{1/2} |f_n - f(x)|\,dx + \underbrace{\int_{1/2}^{1} |1-f(x)|\,dx}_{(2)}$$

if $d(f_n,f)\to 0$, then

(1) $$\int_0^{1/2-1/n} |f(x)|\,dx \to \int_0^{1/2} |f(x)|\,dx = 0 \Rightarrow f(x) = 0$$

(2) $$\int_{1/2}^{1} |1-f(x)|\,dx \to \int_{1/2}^{1} |1-f(x)|\,dx = 0 \Rightarrow 1-f(x) = 0 \Rightarrow f(x) = 1$$

which is a contradiction

Complete Metric Spaces

Definition Complete Metric Spaces

Let (X,d) be a metric space

Then, X is complete $\iff$ any Cauchy sequence point converges to a point in X.

If X is known to be complete and you have a sequence you know is Cauchy then,

$$\exists x \in X \text{ such that } x_n \to x \text{ as } n \to \infty$$

Axiom $\mathbb{R}$ is complete

$\mathbb{Q}$ is not complete

$\frac{p_n}{q_n} \to \sqrt{2}$ in $\mathbb{R}$ $(\sqrt{2} \notin \mathbb{Q})$

put $\mathbb{Q}$ into $\mathbb{R} \rightsquigarrow$ completing $\mathbb{Q}$

Complete is good

<u>Problem</u>: not all metric spaces are complete

<u>Outcome</u>: Complete them and do so systematically

Completion of metric spaces

Theorem Completion of metric space

Let (X,d) be a metric space

Then, there is a metric space $(X^*, \hat{d})$ and and an isometry $\varphi: X \to X^*$ such that

i) X^* is complete

ii) $\varphi(X) = X^*$

We call X^* a completion and all completions of X are isometric to X^*

<u>Proof</u>:

1) <u>CASE 1</u>: X is complete $\implies X = X^*$

2) <u>CASE 2</u>: X is not complete

Let (X,d) be a metric space

Let $\mathcal{C}(X)$ be the set of all Cauchy sequences

Note: $\mathcal{C}(X) \neq \phi$ as $(x, \dots, x) \in \mathcal{C}(X)$

Now define Cauchy sequences (x_n) and (y_n) and a relation on $\mathcal{C}(x)$

$$\forall\ (x_n), (y_n) \in \mathcal{C}(X),\quad (x_n) \sim (y_n) \iff \lim_{n\to\infty} d(x_n, y_n) = 0$$

Claim: '$\sim$' is an equivalence relation.

1) Reflexivity: $(x_n) \sim (x_n)$ since $d(x_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, x_n) = 0$

2) Symmetry: If $(x_n) \sim (y_n)$ then $\lim_{n\to\infty} d(x_n, y_n) = 0 \Rightarrow \lim_{n\to\infty} d(y_n, x_n) = 0$

$$\Rightarrow (y_n) \sim (x_n)$$

3) Transitivity: if $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$ then $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$

By the triangle inequality,

$$0 < d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) \to 0 \text{ as } n \to \infty$$ algebra of limits

$$\Rightarrow \lim_{n\to\infty} d(x_n, y_n) = 0$$

$$\Rightarrow (x_n) \sim (z_n)$$

◀

Thus '$\sim$' is an equivalence relation and $\mathcal{C}(X)$ can be split into equivalence classes.

$$[(x_n)] = \{(a_n) \in \mathcal{C}(X) \mid (x_n) \sim (a_n)\}$$

The following facts are used:

if $(x_n) \in [(a_n)]$ and $(y_n) \in [(a_n)] \Rightarrow (x_n) \sim (y_n)$

if $(x_n) \in [(a_n)]$ and $(y_n) \in [(b_n)] \Rightarrow (x_n) \nsim (y_n)$

Let $\tilde{X}$ be the set of all equivalence classes.

Observe that if $\lim_{n\to\infty} x_n = x$ and $(x_n) \sim (y_n)$, then by triangle inequality

$$0 \leq d(y_n, x) \leq \underbrace{d(y_n, x_n)}_{\to 0 \text{ by } \sim} + \underbrace{d(x_n, x)}_{\to 0}$$

$$\Rightarrow \lim_{n\to\infty} d(y_n, x) = 0 \quad \text{sandwich thm}$$

$$\Rightarrow \lim_{n\to\infty} y_n = x$$

Further observe that if $(x_n) \nsim (y_n)$ then $\lim_{n\to\infty} x_n \neq \lim_{n\to\infty} y_n$.

As if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, then

$$0 \leq d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) \Rightarrow \lim_{n\to\infty} d(x_n, y_n) = 0$$

$$\Rightarrow (x_n) \sim (y_n) \quad ↯ \text{ contradiction}$$

Define function f

$$f: X \to \tilde{X} \; ; \; f(x) = [(x)] \quad \text{where any } (a_n) \in [(x)] \to x \text{ as } n\to\infty$$

Claim: f is one to one

$$f(x) = f(y) \Rightarrow [(x)] = [(y)]$$

$$\Rightarrow (x) \sim (y) \qquad (a \sim b \Leftrightarrow [a] = [b])$$

$$\Rightarrow x = y \qquad \text{(shown above)}$$

◀

Define a metric on $\hat{d}$ on $\tilde{X}$ where

$$\forall [(x_n)], [(y_n)], \quad \hat{d}([(x_n)], [(y_n)]) = \lim_{n\to\infty} d(x_n, y_n) \qquad (x_n) \in [(x_n)]$$

Claim: $\hat{d}$ is a metric $\qquad (y_n) \in [(y_n)]$

M1) $d(x_n, y_n) \geq 0 \Rightarrow \lim_{n\to\infty} d(x_n, y_n) \geq 0 \Rightarrow \hat{d} \geq 0$

M2) $[(x_n)] = [(y_n)] \Rightarrow (x_n) \sim (y_n) \Rightarrow \hat{d}([(x_n)], [(y_n)]) = \lim_{n\to\infty} d(x_n, y_n) = 0$

$\hat{d}([(x_n)], [(y_n)]) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, y_n) = 0 \Rightarrow (x_n) \sim (y_n)$

$$\Rightarrow [(x_n)] = [(y_n)]$$

M3) $d(x_n, y_n) = d(y_n, x_n) \Rightarrow \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n)$

$$\Rightarrow \hat{d}([(x_n)], [(y_n)]) = \hat{d}([(y_n)], [(x_n)])$$

M4) $\hat{d}([(x_n)], [(z_n)]) = \lim_{n\to\infty} d(x_n, z_n)$

$$\leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} (y_n, z_n) \qquad \text{(triangle inequality and algebra of limits)}$$

$$= \hat{d}([(x_n)], [(y_n)]) + \hat{d}([(y_n)], [(z_n)])$$

◀

Claim: $\hat{d}$ is well defined

Suppose $(x_n), (x_n') \in [(x_n)]$ and $(y_n), (y_n') \in [(y_n)]$ such that

$$(x_n) \sim (x_n') \quad \text{and} \quad (y_n) \sim (y_n')$$

$$\Rightarrow \lim_{n\to\infty} d(x_n, x_n') = 0 \quad \text{and} \quad \lim_{n\to\infty} d(y_n, y_n') = 0$$

Now by double triangle inequality

$$d(x_n', y_n') \leq \underbrace{d(x_n', x_n)}_{\to 0} + d(x_n, y_n) + \underbrace{d(y_n, y_n')}_{\to 0}$$

$$\Rightarrow \lim_{n\to\infty} d(x_n', y_n') \leq \lim_{n\to\infty} d(x_n, y_n) \qquad (*1)$$

Similarly,

$$d(x_n, y_n) \leq \underbrace{d(x_n', x_n)}_{\to 0} + d(x_n', y_n') + \underbrace{d(y_n, y_n')}_{\to 0}$$

$$\Rightarrow \lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} d(x_n', y_n') \qquad (*2)$$

From inequalities $(*1)$ and $(*2)$

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n', y_n')$$

hence well-defined

◀

Claim: f is an isometry

Let (X, d) and $(\tilde{X}, \hat{d})$ be metric spaces.

$f: X \to \tilde{X}$, $f(x) = [(x)]$ where any $(a_n) \in [(x)] \to x$ as $n \to \infty$

$$\begin{aligned}\hat{d}(f(x), f(y)) &= \hat{d}([(x)], [(y)]) \\ &= \lim_{n\to\infty} d(x, y) \\ &= d(x, y)\end{aligned}$$

◀

Claim: The limit defining $\hat{d}$ exists

Let (x_n) and (y_n) be Cauchy sequences in X. By the triangle inequality,

$$\begin{aligned}|d(x_n, y_n) - d(x_m, y_m)| &= |d(x_n, y_n) - d(x_n, y_m) + d(x_n, y_m) - d(x_m, y_m)| \\ &\leq |d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)| \quad \Delta\text{-ineq} \\ \text{rearrangement of } \Delta\text{-inequality} \quad &\leq d(y_n, y_m) + d(x_n, x_m)\end{aligned}$$

Since (x_n) is Cauchy, let $d(x_n, x_m) < \varepsilon/2$

Since (y_n) is Cauchy, let $d(y_n, y_m) < \varepsilon/2$

Therefore $|d(x_n, y_n) - d(x_m, y_m)| < \varepsilon \Rightarrow (d(x_n, y_n))_{n\in\mathbb{N}}$ is Cauchy

$(d(x_n, y_n))_{n\in\mathbb{N}}$ is a sequence of real numbers and Cauchy $\Longrightarrow \lim_{n\to\infty} d(x_n, y_n)$ exists

Completeness in $\mathbb{R}$
All cauchy sequences converge

Claim: $\overline{f(X)} = \tilde{X}$ (density)

Suppose (x_n) is a Cauchy sequence and $(x_n) \in [(x_n)]$. By defn of Cauchy,

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \text{ such that } \forall m, n > N, \; d(x_m, x_n) < \varepsilon = \frac{1}{K}$$

Let $m = n_K$, then $d(x_{n_K}, x_n) < \frac{1}{K}$

Further, let $[(x_{n_K})]$ be an equivalence class containing all Cauchy sequence converging to x_{n_K},

$$[(x_{n_K})] = f(x_{n_K}) \qquad (\text{any } (a_n) \in [(x_{n_K})] \to x_{n_K})$$

Then

$$\hat{d}([(x_n)], f(x_{n_K})) = \hat{d}([(x_n)], [(x_{n_K})]) = \lim_{n\to\infty} d(x_n, x_{n_K}) \leq \frac{1}{K}$$

$$\Longrightarrow [(x_n)] = \lim_{K\to\infty} f(x_{n_K})$$

Therefore for any $\varepsilon = 1/K > 0$ and any $[(x_n)] \in \tilde{X}$, $\exists\, \alpha = f(x_{n_K}) = [(y_{n_K})]$ such that

$$\hat{d}([(x_n)], f(x_{n_K})) < 1/K$$

i.e. every $[(x_n)] \in X$ is the limit of a sequence in $f(X)$

$$\Rightarrow \tilde{X} = \overline{f(X)}$$

Now we show that $\tilde{X}$ is complete.

Showing that any Cauchy sequence in $\tilde{X}$ converges to a point in $\tilde{X}$

Consider a Cauchy sequence in $\tilde{X}$

$$([x_n^{(K)}])_{K\in\mathbb{N}} = ([(x_n^{(1)})], [(x_n^{(2)})], [(x_n^{(3)})], \dots)$$

where $[(x_n^{(K)})]$ be the K^{th} Cauchy sequence in $\tilde{X}$

Let $P_K = [(x_n^{(K)})]$ for $K \in \mathbb{N}$

By density, for a fixed K, each P_K is a limit of some $[(y_n^{(K)})] \in f(X)$, i.e.

$$\hat{d}(p_K, q_K) < \frac{1}{K}$$

where $q_K = [(y_n^{(K)})]$

The sequence (q_K) can be shown to be Cauchy as follows

$$\begin{aligned}\hat{d}(q_K, q_\ell) &\leq \hat{d}(q_K, p_K) + \hat{d}(p_K, q_\ell) \\ &\leq \hat{d}(q_K, p_K) + \hat{d}(p_K, p_\ell) + \hat{d}(p_\ell, q_\ell) \\ &\leq \frac{1}{K} + \frac{1}{\ell} + \hat{d}(p_K, p_\ell)\end{aligned}$$

(p_K) is Cauchy so we can choose K, ℓ as large as we like making RHS as small as we like

Since $q_K \in f(X)$, $\exists\, y_K \in X$ such that

$$f(y_K) = q_K = [(y_n^K)] \quad \text{for a fixed } K.$$

The sequence (y_K) must be Cauchy as $([(y_n^K)])_{K\in\mathbb{N}}$ is Cauchy in $\tilde{X}$ and f is isometric

$$\hat{d}(f(y_K), f(y_\ell)) = \hat{d}(q_K, q_\ell) = d(y_K, y_\ell) \Longrightarrow \text{Cauchy}$$

Therefore (y_K) belongs to some equivalence class $[(x_n)] \in \tilde{X}$

Claim: $\lim\limits_{K\to\infty} \hat{d}([(x_n^{(K)})], [(x_n)]) = 0$

Take any $\varepsilon > 0$ and observe that

$$\begin{aligned}\hat{d}([(x_n^{(K)})], [(x_n)]) &\leq \hat{d}([(x_n^{(K)})], [(y_n^{(K)})]) + \hat{d}([(y_n^{(K)})], [(x_n)]) \quad \text{triangle inequality} \\ &\leq \hat{d}(p_K, q_K) + \hat{d}([(y_n^{(K)})], [(x_n)]) \\ &< \frac{1}{K} + \hat{d}([(y_n^{(K)})], [(x_n)])\end{aligned}$$

$$\hat{d}([(y_n^{(K)})], [(x_n)]) = \hat{d}(f(y_K), [(x_n)]) = \lim_{K\to\infty} d(y_K, y_n) \leq \varepsilon$$

$$((y_n) \in [(x_n)])$$

for sufficiently large K since (y_K) is Cauchy in X.

Therefore $\lim\limits_{K\to\infty} \hat{d}([(y_n^{(K)})], [(x_n)]) = 0$ and since $1/K \to 0$ as $K \to \infty$, we get

$$\lim_{K\to\infty} \hat{d}([(x_n^{(K)})], [(x_n)]) = 0$$

and therefore $\tilde{X}$ is complete

Uniqueness: Suppose that

$$(X^*, d^*) \quad \text{and} \quad (X^{**}, d^{**})$$

are two completions. Need to show that these are equivalent, i.e. isometric.

Consider any arbitrary $x^* \in X^*$. Since X^* is a completion, there is a Cauchy sequence (x_n) in X such that

$$x_n \to x^* \text{ as } n \to \infty$$

Similarly, assume (x_n) belongs to X^{**}. Since X^{**} is complete,

$$x_n \to x^{**} \quad \text{as} \quad n \to \infty$$

where $x^{**} \in X^{**}$

Define function

$$\varphi : X^* \to X^{**} ; \ \varphi(x^*) = x^{**}$$

Claim: φ is one to one

Since X^* is complete, $\exists$ Cauchy sequences in X (x_{1n}) and (x_{2n}) such that

$$x_{1n} \to x_1^* \quad \text{and} \quad x_{2n} \to x_2^* \quad \text{as} \quad n \to \infty$$

Suppose $f(x_1^*) = f(x_2^*) \implies x_1^{**} = x_2^{**}$

Therefore there is a Cauchy sequence in X, (x_{1n}^{**}) and (x_{2n}^{**}) s.t

$$x_{1n}^{**} \to x_1^{**} \quad \text{and} \quad x_{2n}^{**} \to x_2^{**}$$

Since $x_1^{**} = x_2^{**}$, $\lim_{n \to \infty} d(x_{1n}^{**}, x_{2n}^{**}) = 0$

Since X^* is complete, $x_{1n}^{**} \to x_1^*$ and $x_{2n}^{**} \to x_2^*$ as $n \to \infty$ in X^*

Therefore since

$$\lim_{n \to \infty} d(x_{1n}^{**}, x_{2n}^{**}) = 0 \implies d(x_1^*, x_2^*) = 0 \implies x_1 = x_2$$

From above, φ does not depend on choice of sequence of $(x_n)_{n \geq 1}$

Claim: For $x \in X$, $\varphi(x) = x$

If $x \in X$, then the constant sequence

$$(x, x, \cdots, x)$$

is a sequence in X^* which converges to x. So $f(x)$ is the limit in Z of $(x, \cdots, x)$ which is x

$$\implies f(x) = x$$

Further, $\forall\, x_1^*, x_2^* \in X$,

$$d^{**}(x_1^{**}, x_2^{**}) = d^{**}(\varphi(x_1^*), \varphi(x_2^*)) = d^*(x_1^*, x_2^*)$$

hence isometric ■

Examples of complete metric spaces

Proposition

The metric space $(\mathbb{R}^N, d_\infty)$ with

$$d_\infty(\underline{x}, \underline{y}) = \sup\{|x_i - y_i| : 1 \le i \le N\}$$

is a complete metric space

Proof: Take a Cauchy sequence in $\mathbb{R}^N$

$$(\underline{x}_n)_{n=1}^{\infty}$$

Recall the notation $\underline{x}_n = (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})$

By defn of Cauchy, given $\varepsilon > 0$, $\exists\, N = N(\varepsilon) > 0$ such that $\forall\, m, n > N$,

$$d_\infty(\underline{x}_n, \underline{x}_m) < \varepsilon \implies \max\{|x_n^{(i)} - x_m^{(i)}| : 1 \le i \le N\} < \varepsilon$$

$$\implies |x_n^{(i)} - x_m^{(i)}| < \varepsilon \quad \text{for each } i$$

Therefore sequence of real numbers $(x_n^{(i)})_{n=1}^{\infty}$ is Cauchy for each i.

Since $\mathbb{R}$ is complete, $(x_n^{(i)})$ converges: $\exists\, x_i \in \mathbb{R}$ such that

$$\lim_{n\to\infty} x_n^{(i)} = x_i$$

$$\underline{x}_1 = (x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, \dots, x_1^{(N)})$$
$$\underline{x}_2 = (x_2^{(1)}, x_2^{(2)}, x_2^{(3)}, \dots, x_2^{(N)})$$
$$\underline{x}_3 = (x_3^{(1)}, x_3^{(2)}, x_3^{(3)}, \dots, x_3^{(N)})$$
$$\vdots$$

N

$$\downarrow \quad \downarrow \quad \downarrow, \dots, \downarrow$$
$$\underline{x} = (x_1, x_2, x_3, \dots, x_N)$$

Construct candidate limit $\underline{x} = (x_1, \dots, x_n)$

Recall in $(\mathbb{R}^N, d_\infty)$, the sequence $(\underline{x_N})_{n=1}^{\infty}$ converges to $\underline{x} \in (x_1, \dots, x_N)$

$$\Updownarrow$$

$(x_n^{(i)})_{n=1}^{\infty}$ converges to x_i (true by completeness)

$\therefore$ Cauchy sequence $(x_n)_{n=1}^{\infty} \in (\mathbb{R}^N)^{\mathbb{N}}$ converges in $\mathbb{R}^N$

$\Rightarrow (\mathbb{R}^N, d_\infty$ is complete

$\blacksquare$

Proposition:

The metric space (X, d_p) with $X = \mathbb{R}^n$

$$d_p(\underline{x}, \underline{y}) = \left(\sum_{i=1}^{n} |x_i - y_i| \right)^{1/p}, \quad p \geq 1$$

is a complete metric space

Proof:

Let $\{\underline{x_m}\}_{m \geq 1}$ be any arbitrary Cauchy sequence in $(\mathbb{R}^n, d_p)$ where

$$\underline{x}_m = (x_m^{(1)}, x_m^{(2)}, \dots, x_m^{(n)})$$

Since $\{\underline{x}_m\}_{m \geq 1}$ is Cauchy, given $\varepsilon > 0$, $\exists N = N_\varepsilon \in \mathbb{N}$ s.t

$$d_p(\underline{x}_m, \underline{x}_n) = \left(\sum_{i=1}^{n} |x_m^{(i)} - x_n^{(i)}|^p \right)^{1/p} < \varepsilon \quad \text{for all } m, n > N_\varepsilon$$

$$\Rightarrow \quad |x_m^{(i)} - x_n^{(i)}| < \varepsilon \quad \forall n, m > N_\varepsilon.$$

Therefore the sequence

$$\{\underline{x}_m^{(i)}\}_{m=1}^{\infty}$$

is Cauchy and by completeness of $\mathbb{R}$, it converges

$$\lim_{m \to \infty} x_m^{(i)} = x_i$$

Therefore construct candidate limit

$$\underline{x} = (x_1, x_2, \dots, x_n) \quad \text{candidate limit}$$

It is obvious that $\underline{x} \in \mathbb{R}^n$

Just need to show that $\{\underline{x}_m\} \to \underline{x}$ as $m \to \infty$

$$d_p(\underline{x}_m, \underline{x}_n) = \left(\sum_{i=1}^{n} |x_m^{(i)} - x_n^{(i)}|^p\right)^{1/p} < \varepsilon \implies \sum_{i=1}^{n} |x_m^{(i)} - x_n^{(i)}|^p < \varepsilon^p \quad (*)$$

Let $n \to \infty$, we get (by completeness, $x_m^{(i)} \to x_i$)

$$\sum_{i=1}^{n} |x_m^{(i)} - x_i|^p < \varepsilon^p \implies d_p(\underline{x}_m, \underline{x}) < \varepsilon$$

$$\implies \underline{x}_n \to \underline{x} \text{ as } n \to \infty$$

Hence Cauchy sequence $\{\underline{x}_m\}$ converges in $\mathbb{R}^n$

$$\implies (\mathbb{R}^n, d_p) \text{ is complete.}$$

Space of bounded functions are complete

Proposition Space of bounded functions is complete

The space of bounded functions real valued functions $B(S)$ is complete under uniform metric d_∞

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in S\}$$

i.e. $(B(S), d_\infty)$ is complete

Proof: Consider any Cauchy sequence

$$(f_n)_{n=1}^{\infty}$$

By definition of Cauchy, $\forall \varepsilon > 0$, $\exists\, N = N(\varepsilon)$ s.t $\forall\, m, n \geq N$,

$$d_\infty(f_n, f_m) < \varepsilon$$

$$\implies \sup_{x \in S} |f_n(x) - f_m(x)| < \varepsilon \implies |f_m(x) - f_n(x)| < \varepsilon$$

So the sequence of real numbers $(f_n(x))_{n=1}^{\infty}$ is Cauchy

Since $\mathbb{R}$ is complete,

$$f_n(x) \to f_x \quad \text{as} \quad n \to \infty$$

Candidate limit

$$f : S \to \mathbb{R};\ f(x) = f_x$$

Showing that

1) $f_n \to f$ as $n \to \infty$

2) f is bounded $\Rightarrow f \in B(S)$

2) f is bounded.

a) Since $f_n(x)$ is Cauchy $\Rightarrow$ convergent

$$|f_n(x) - f(x)| < \varepsilon$$

b) Since f_n is bounded,

$$|f_n| < M \quad \text{for some } M \in \mathbb{R}$$

$$\begin{aligned} |f(t)| &= |f(t) - f_n(t) + f_n(t)| \\ &\le |f(t) - f_n(t)| + |f_n(t)| \\ &< \varepsilon + R \end{aligned}$$

$\Rightarrow |f(t)| < \varepsilon + R$

$\Rightarrow f$ is bounded $\Rightarrow f \in B(S)$

1) Showing that $f_n \to f$ uniformly

We know $f_n(x)$ is Cauchy $\Rightarrow |f_n(x) - f_m(x)| < \varepsilon$

Consider

$$\begin{aligned} |f_n(t) - f(t)| &= |f_n(t) - f_m(t) + f_m(t) - f(t)| \\ &\le |f_n(t) - f_n(t)| + |f_m(t) - f(t)| \end{aligned}$$

Since $f_m(t) \to f_t = f(t)$, pointwise $\Rightarrow |f_m(t) - f(t)| \to 0$ as $m \to \infty$

$\Rightarrow |f_n(t) - f(t)| \le \varepsilon \quad \forall n > N$ and all $t \in S$

$\Rightarrow \quad d_\infty(f_n, f) \le \varepsilon$

$\Rightarrow f_n \to f$ ■

Lemma

Consider (X, d_0) where

$$d_0(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \qquad \text{discrete metric}$$

For any sequence $(x_n)_{n=1}^{\infty}$, if (x_n) converges, then it is eventually constant

<u>Proof</u>: Suppose that

$$\lim_{n\to\infty} x_n = x$$

By definition of convergence

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \text{ s.t } \forall n > N,$$

$$d_0(x_n, x) < \varepsilon$$

Set $\varepsilon = 1/2 \implies \exists N_0 = N(1/2)$ such that

$$d_0(x_n, x) < \frac{1}{2} \quad \forall n \geq N_0 \implies d_0(x_n, x) = 0 \qquad \text{by definition of discrete metric}$$

$$\implies x_n = x \quad \forall n \geq N_0$$

$$\implies \text{eventually constant} \qquad \blacksquare$$

Discrete metric is complete

Proposition

Metric space (X, d_0) with

$$d_0(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

is complete

<u>Proof</u>: Consider any Cauchy sequence

$$(x_n)_{n=1}^{\infty}$$

By definition of Cauchy,

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \text{ s.t } \forall m, n > N$$

$$d_0(x_m, x_n) < \varepsilon$$

Set $\varepsilon = 1/2 \implies \exists N_0 = N(1/2)$ such that

$$d_0(x_m, x_n) < \frac{1}{2} \quad \forall n \geq N_0 \implies d_0(x_m, x_n) = 0 \qquad \text{by definition of discrete metric}$$

$$\implies x_n = x_m \quad \forall n \geq N_0$$

Hence sequence eventually constant $\Rightarrow$ Cauchy sequence $(x_n)_{n=1}^{\infty}$ converges

$\Rightarrow (X, d_0)$ is complete ■

Basic steps to show space is Complete

1) To show a metric space (X,d) is complete
   - Start with an arbitrary Cauchy sequence $(x_n)_{n=1}^{\infty}$
   - construct a candidate limit x using definition of Cauchy under d metric
   - show that $x_n \to x$ as $n \to \infty$, $x_n \in X^{\mathbb{N}}$
   - Show that $x \in X$

2) To show a metric space is not complete, find one Cauchy sequence that does not converge to a point in space

Some properties of Complete spaces

Theorem

Let (X,d) be a metric space

Let A be a non-empty subset of X, i.e. $A \subseteq X$, $A \neq \emptyset$ so (A,d) is a metric space

Then

i) if (A,d) is complete $\Rightarrow$ A is closed in X

ii) If X is complete and A is closed in (X,d) then (A,d) is complete.

Proof:

i) By definition of complete

A is complete $\Leftrightarrow$ every Cauchy sequence converges to a point in A

Suffices to show that $A' \subseteq A$

Suppose that $x \in A'$. Then there is a convergent sequence $(x_n)_{n=1}^{\infty}$ such that

$$x_n \to x \quad \text{as} \quad n \to \infty$$

But convergent sequence $\Rightarrow$ Cauchy sequence and therefore (x_n) is Cauchy

Therefore by the definition of completeness, $x \in A$. We have that

$$x \in A' \Rightarrow x \in A$$

And therefore

$$A' \subseteq A \implies A \text{ is closed.}$$

(ii) Let $(x_n)_{n=1}^{\infty}$ be a Cauchy Sequence in A

Since $(x_n)_{n=1}^{\infty}$ is Cauchy in A and $A \subseteq X$, $(x_n)_{n=1}^{\infty}$ is Cauchy in X.

Therefore by completeness in X,

$$x_n \to x \in X \quad \text{as } n \to \infty$$

But as A is closed $\Rightarrow$ A contains all its limit points (proved in Lecture 7)

$\Rightarrow$ all Cauchy sequences $(x_n)_{n=1}^{\infty}$ converge to a point in A

$\Rightarrow$ A is complete.

$\blacksquare$